THE COMPOSITE FERMION THEORY OF FRACTIONAL QUANTUM HALL EFFECT

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Abstract

This report gives an overview of the integer and fractional quantum Hall effects, beginning with Landau level physics and the role of disorder. It then shows how strong interactions produce fractional states, and discusses both Laughlin's theory and the composite fermion framework. Finally, it highlights how these two approaches are closely connected.

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List of Abbreviations

Abbreviation	Full Form
2DEG	Two-Dimensional Electron Gas
QHE	Quantum Hall Effect
IQHE	Integer Quantum Hall Effect
FQHE	Fractional Quantum Hall Effect
LL	Landau Level
LLL	Lowest Landau Level
CF	Composite Fermion

1 Introduction

The Quantum Hall Effect (QHE) is one of the most important discoveries in condensed matter physics because it shows how a simple two–dimensional electron system can give rise to quantization without confinement and new phases of matter when placed in a strong magnetic field. Over the years, it has become clear that the QHE is not just a transport phenomenon but possess topological phases, and emergent quasiparticles.

The first step in understanding this subject is the Integer Quantum Hall Effect (IQHE), discovered by von Klitzing in 1980. The IQHE can be explained using single–particle physics. Electrons move in quantized cyclotron orbits forming Landau levels, and because of disorder these levels broaden but still produce plateaus in the Hall conductance. Since the explanation does not require interactions, IQHE serves as a starting point for the more complicated fractional case.

The situation becomes far richer with the discovery of the Fractional Quantum Hall Effect (FQHE) by Tsui, Störmer and Laughlin in 1982. It has been understood that the fractional plateaus cannot be explained without electron–electron interactions, which play a central role. Laughlin proposed a many–body wave-function for filling factors $\nu = 1/m$ (with m odd), which successfully explains $1/m$ fractional plateau.

After the Laughlin state, several approaches attempted to extend the description to other fractional filling factors. Several hierarchical models developed by Haldane, Halperin and others suggested that new states may form by condensing the quasiparticles or quasi-holes of an existing fractional state. Around the same time, Chern–Simons effective field theories were introduced as a way to describe the low–energy physics of the fractional states.

A major advancement occurred in 1989, when Jain introduced the idea of composite fermions. The basic idea is of composite fermion is each electron binds an even number of magnetic flux quanta, and the resulting weakly interacting composite fermion moves in a reduced effective magnetic field. Under this effective field, composite fermions can fill integer numbers of Landau levels, making many fractional quantum Hall states appear as the integer quantum Hall effect of composite fermions. This picture not only explains a large set of experimentally observed fractions but also provides a clear physical intuition for why these states form.

The report is organized as follows. In Sec. 2, we introduce the basic overview of IQHE, then we develop the theory of FQHE through Laughlin's approach in Sec. 3, and finally in Sec. 4, we explain how the composite fermion picture offers a unified way of understanding many fractional quantum Hall states and its connection with Laughlin's idea of FQHE.

2 Integer Quantum Hall Effect

The Integer Quantum Hall Effect (IQHE) was discovered by von Klitzing in 1980. It arises in a two–dimensional electron gas when a strong magnetic field is applied perpendicular to the plane. In the standard configuration, with $\vec{B} = B\hat{z}$ and an in–plane electric field $\vec{E} = E\hat{x}$, the measured resistivity tensor shows two remarkable features:

$$\rho_{xx} = 0, \qquad \rho_{xy} = \frac{2\pi\hbar}{e^2\,\nu}, \tag{2.1}$$

where $\nu \in \mathbb{N}$ is the integer filling factor of the Landau levels. The vanishing longitudinal resistivity, along with the robust quantization of Hall resistivity ρ_{xy}, as shown in figure 1, is one of the fundamental signatures of the IQHE.

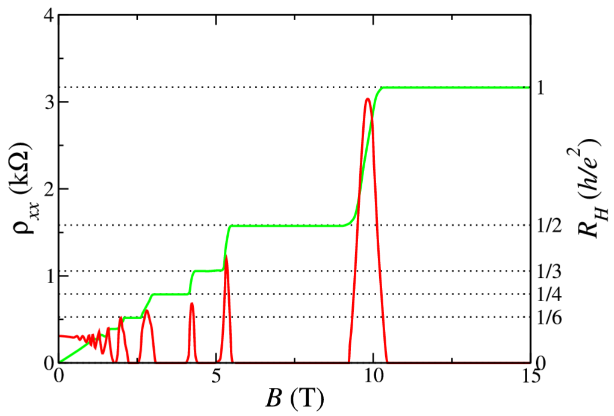


Figure 1: Figure shows experimental result of longitudinal resistivity (in red line) and Hall resistivity (in green line) of a 2D quantum hall system of a low purity material (usually GaAs).

2.1 Quantized Transport and the Role of Disorder

A striking aspect of the integer effect is its insensitivity to impurities and imperfections in the sample. Real materials always contain disorder, and rather than destroying quantization, the presence of disorder actually helps stabilize it. Due to presence of disorder ($V_{\text{disorder}} \ll \hbar\omega_B$), each Landau level is broadened into a energy band since degeneracy is lifted. But most of the states in this band become localized in the bulk due to the random potential landscape. Only a small set of states corresponding to the centre of the energy band remain extended (as shown in figure 2). The confining potential at the sample boundary gives rise to those extended state as shown in the figure 3b. Notably, only the delocalized states carry currents, thus contribute to conductivity.

For the quantum Hall system with electron density n and magnetic field $B = n\Phi_0/\nu$ one expects the Hall resistivity to be

$$\rho_{xy} = \frac{2\pi\hbar}{e^2\,\nu}. \tag{2.2}$$

Since there are very few delocalized states present in each LL, for $B \neq n\Phi_0/\nu$, the Hall resistivity remain fixed to the previous value. Consequently, the disorder within the system provides the stability of the Hall conductivity.

Moreover the extended state in the sample boundary propagate in a single direction. Since back-scattering is forbidden for a chiral channel, these edge modes carry current robustly. We call these states as *chiral* edge states. One can compare these chiral edge states with the classical skipping orbit of the electron in the edge of the sample of Hall setup. This reflects the topological characteristics of the IQHE.

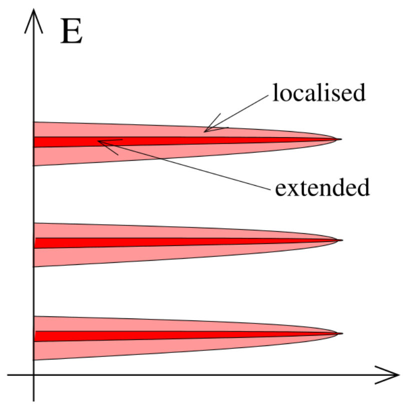


Figure 2: Figure shows the broadening of energy bands in Landau levels due to presence of disorder. The extended states (in saturated red) appear at the centre of the energy bands surrounded by the localized states (in light red).

2.2 Landau Level physics

To explore the structure of the electronic states in a typical quantum Hall system, one can choose a simpler gauge, the Landau gauge $\vec{A} = (0,\, xB)$, and expect a quantized Landau levels (LL) energy spectrum i.e.,

$$E_n = \hbar\omega_B\left(n + \frac{1}{2}\right), \quad n \in \mathbb{N} \tag{2.3}$$

where $\omega_B = eB/m$ is *cyclotron frequency* of electron. Further calculation showed that each LLs are highly degenerate, such that the degeneracy of each LL per unit area is

$$G = \frac{N}{A} = \frac{B}{\Phi_0} \tag{2.4}$$

where B is external magnetic field and $\Phi_0 = e/2\pi\hbar$ is *flux quantum*. Thus the filling factor is given by

$$\nu = \frac{\rho}{G} = \frac{\rho\Phi_0}{B} \tag{2.5}$$

where ρ is the 2DEG density. It is clear that as the magnetic field is increased, each Landau level can accommodate more and more electrons, and, as a result, fewer and fewer Landau levels are occupied.

Considering the relevance to the FQHE, we adopt the symmetric gauge,

$$\vec{A} = \frac{1}{2}(-yB,\, xB), \tag{2.6}$$

which makes the rotational symmetry of the problem explicit. In this gauge, the LLs are given by the lowest Landau level (LLL) wave-functions are given by

$$\psi_{0,m}(z) = \psi_{LLL,m}(z) = \left(\frac{z}{l_B}\right)^m e^{-|z|^2/4l_B^2}, \qquad z = x - \iota y, \tag{2.7}$$

where $m \in \mathbb{Z}_{\geq 0}$ reflects the vast degeneracy of the LLs and $l_B = \sqrt{\hbar/eB}$ is *magnetic length*. One can obtain the higher LL wave functions by using the raising operator:

$$\psi_{n,m} \sim \left(l_B\partial - \frac{\bar{z}}{4l_B}\right)^n \psi_{0,m}(z), \quad \partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}\right) \tag{2.8}$$

Within the Lowest Landau Level (LLL), an alternative interpretation of the quantum number m reveals it as the angular momentum, which can be seen from $\psi_{LLL,m}$ as eigenstates of the angular momentum operator,

$$\hat{J}\psi_{LLL,m}(z) = i\hbar(z\partial - \bar{z}\bar{\partial})\psi_{LLL,m}(z) = m\hbar\ \psi_{LLL,m}(z) \tag{2.9}$$

where $\bar{z}$ and $\bar{\partial}$ denote the complex conjugate of z and ∂ respectively. Further the profile of the LLL wave-functions $\psi_{LLL,m}$ are similar to concentric diffused rings of thickness of order L_B and radius of the peak of the diffused rings associated with the m-th orbital is $r_m = \sqrt{2m}\, l_B$.

It follows from the above discussion that the degeneracy of a Landau level is therefore determined by the number of allowed angular momentum values within the finite sample area. The wave-function with angular momentum m is peaked on a ring of radius $r_m = \sqrt{2m}\, l_B$. Let in a disc shaped region of area $A = \pi R^2$, the maximum number of electron is N, thus the wave-function with maximum radius is $r_N = \sqrt{2N}\, l_B$. Then the degeneracy of LLL i.e., N is given by

$$\begin{aligned} r_N^2 &= R^2 \\ N &= \frac{R^2}{2l_B^2} = \frac{eAB}{2\pi\hbar} = \frac{\Phi}{\Phi_0} \end{aligned} \tag{2.10}$$

where Φ is total flux associated with external magnetic field. The obtained result is consistent with the known result derived using the Landau gauge.

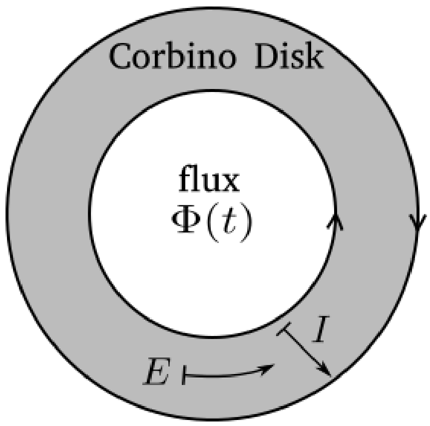
For many non–interacting electrons as it is assumed while explaining the IQHE, the ground state of fully filled LLL is formed by occupying all the N available angular momentum states. The resulting wave-function is a Slater determinant built from the single–particle states. This can be written as the *Vandermonde determinant*:

$$\psi_{LLL} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} e^{-\sum_i |z_i|^2/4l_B^2} = \prod_{i<j}(z_j - z_i)\, e^{-\sum_i |z_i|^2/4l_B^2}. \tag{2.11}$$

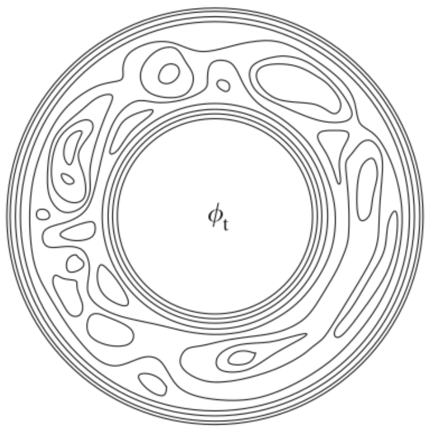
This wave-function describes a completely filled LLL and is the starting point for understanding the electron electron interaction in fractional Hall states.

2.3 Corbino Geometry and Spectral Flow

Another useful viewpoint on the IQHE comes from the Corbino disc geometry. In this setup, we apply a perpendicular magnetic field B. But instead of applying a voltage along the sample edges, one threads an additional magnetic flux through the hole at the centre of the disc as shown in the figure 3a. If the flux is increased adiabatically from 0 to one flux



(a)



(b)

Figure 3: Figure (a) shows the Corbino disc geometry. The time varying flux is applied through the centre of the disc. Figure (b) shows the the extended and localized states in the Corbino disc. One can observe most of the extended states are near the edge of the sample forms a concentric ring, as expected, since there is no disorder effect in the edge. The localized states are in the bulk of the disc due to the presence of strong disorder.

quantum (Φ_0), according to Faraday's law, it produces an azimuthal electric field, which in turn generates a radial current through the sample. For each flux quantum passing through the hole, exactly one electron is transferred from inner circle to outer circle for each LL. From the charge transferred during this process, one can extract the Hall conductivity for each LL, which turns out to be

$$\sigma_{xy} = \frac{e}{\Phi_0}, \tag{2.12}$$

and for ν filled LL it gives the quantized Hall conductivity, i.e.,

$$\sigma_{xy} = \nu \frac{e}{\Phi_0}. \tag{2.13}$$

Moreover, the single charge transfer due to each flux quantum insertion is the result of *spectral flow* of single particle state. If before flux insertion i.e., $\Phi = 0$ the wave-function is $\psi_{LLL,m}(\Phi = 0)$, as the flux is increased by one quantum, each LLL orbital shifts by one unit of angular momentum:

$$\psi_{LLL,m}(\Phi = \Phi_0) = \psi_{LLL,m+1}(\Phi = 0). \tag{2.14}$$

This shift reflects the topological structure of the Landau levels and provides a clean, gauge invariant way of understanding the quantization of the Hall conductance. Altogether, the IQHE combines simple Landau level physics with topological robustness, forming the basis upon which the fractional effect and the composite fermion picture are built.

3 Fractional Quantum Hall Effect

The discovery of the Fractional Quantum Hall Effect (FQHE) in 1982 revealed that the quantization of the Hall conductance is not restricted to integer values alone. Two–dimensional electron systems were found to exhibit clear Hall plateaus at fractional filling factors such as $\nu = 1/3, 2/5, 3/7$, and so on. A striking pattern was that most experimentally observed fractions have odd denominators.

The appearance of these fractional plateaus cannot be understood within a non–interacting picture. Unlike the integer effect, where Landau quantization and disorder are sufficient, the fractional states require a collective behaviour of electrons.

3.1 Importance of Electron–Electron Interaction

The need for interactions becomes especially evident when one examines the structure of the lowest Landau level (LLL). Keeping aside the role of disorder, a necessary condition for the observation of quantized Hall conductivity, as demonstrated in the IQHE, is the existence of quantized energy levels (Landau levels). Disorder of the system, on top of that, provides the stability and robustness of the Hall plateaus. Now to get a plateau at fractional fillings, for example at $\nu = 1/3$, we need a energy level and a energy gap in the spectrum at $\nu < 1$ filling of LLL (as shown in figure 5) even without considering the disorder. This is possible when the Coulomb interaction mixes states within the same Landau level. Importantly, this mixing cannot be treated perturbatively. Thus energy gaps observed at fractional

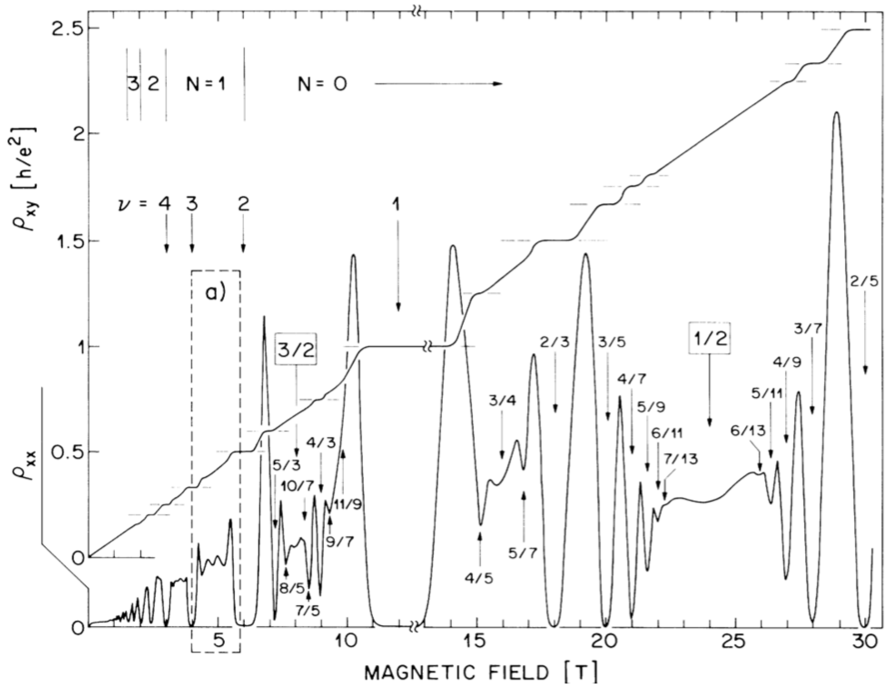


Figure 4: Figure shows the fractional plateau of Hall resistivity along with the longitudinal resistivity. The dip in longitudinal resistivity appear when there is a jump in Hall resistivity plateau. In higher magnetic field, we expect 1/3 fractional plateau which is not shown in the plot.

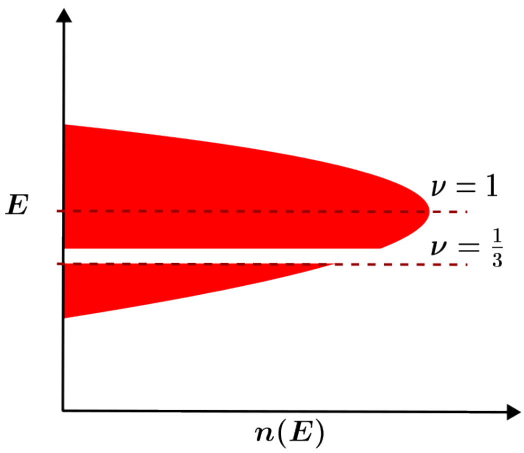


Figure 5: Figure shows the schematic of the LLL in FQHE. The broadening of the LLL and the gap in spectrum for $\nu = 1/3$ is due to mixing of LLs.

fillings arise from the electron– electron interaction and not from single–particle effects.

While localization due to disorder plays a useful role in the integer case, it is not sufficient to explain the fractional plateaus. Experiments show that in very clean samples, especially at small filling factors, transitions between integer plateaus do not occur directly; instead, intermediate correlated states appear. Moreover, in weakly disordered systems, the behaviour deviates from what one would expect if localization were the dominant effect. These observations make it clear that interactions plays a major role in the formation of fractional quantum Hall states.

3.2 Laughlin's Approach

A systematic theoretical description of the FQHE began with Laughlin's proposal for the $\nu = 1/m$ states, where m is an odd integer. Starting from the full Hamiltonian of electrons in a magnetic field,

$$\mathcal{H} = \sum_j \frac{1}{2m_b}\left[\frac{\hbar}{i}\nabla_j + \frac{e}{c}\mathbf{A}(\mathbf{r}_j)\right]^2 + \frac{e^2}{\epsilon}\sum_{j<k}\frac{1}{|\mathbf{r}_j - \mathbf{r}_k|} + \sum_j V_{\text{disorder}}(\mathbf{r}_j) + g\mu_B \mathbf{B}\cdot\mathbf{S}, \tag{3.1}$$

one can simplify the problem by focusing on the no disorder limit and assuming fully spin–polarized electrons restricted to the lowest Landau level. Following the framework explained in IQHE, we can always introduce the disorder later to explain the stability of fractional plateaus. Now the idealized Hamiltonian is

$$\mathcal{H} = \mathcal{P}_{LLL}\frac{e^2}{\epsilon}\sum_{j<k}\frac{1}{|\mathbf{r}_j - \mathbf{r}_k|}\mathcal{P}_{LLL}, \tag{3.2}$$

where $\mathcal{P}_{LLL}$ is LLL projection operator. Crucially, the Coulomb interaction cannot be treated as a small perturbation as there is no tunable parameter.

Laughlin's key insight was to propose an explicit form for the ground–state wave-function rather than attempting to solve the interacting Hamiltonian directly. His construction begins with an ansatz, following from Eq. (2.11),

$$\psi_m(\{z_i\}) = F(\{z_i\})\, e^{-\sum_i |z_i|^2/4l_B^2}, \tag{3.3}$$

where F is a holomorphic, antisymmetric polynomial. Assuming F factorizes pairwise, motivated by Jastrow-type variational wave functions for superfluid Helium, one can write

$$F(\{z_i\}) = \prod_{i<j} f(z_i - z_j), \tag{3.4}$$

with several consistency requirements narrow down the form of $f(z)$. Fermion statistics demands anti-symmetry of the wave-functions i.e., $f(-z) = -f(z)$. For central potential, full wave function is an eigenstate of the total angular momentum. Therefore the rotational invariance requires that under $z \mapsto ze^{\imath\theta}$, the function transforms as $f(z) \mapsto e^{\imath m\theta} f(z)$ for some integer m. These conditions lead naturally to the choice

$$f(z) = z^m, \tag{3.5}$$

with m being odd integer to ensure anti-symmetry. The resulting Laughlin wave-function is

$$\psi_m(\{z_i\}) = \prod_{i<j}(z_i - z_j)^m \exp\left(-\sum_{i=1}^{N} \frac{|z_i|^2}{4l_B^2}\right), \tag{3.6}$$

a remarkably simple expression capturing the essential correlations of the $\nu = 1/m$ states. It is seen numerically that, for small numbers of particles, this wave-function has greater than 99% overlap with the true ground state of the system. This high numerical overlap between the Laughlin wave-function and the system's true ground state provide a strong evidence validating Laughlin's framework.

The factor m determines the extent of repulsion between electrons and also fixes the filling factor. Since the degeneracy of the LLL in this picture is effectively mN, the filling factor becomes

$$\nu = \frac{1}{m}.$$

While the Laughlin state beautifully explains the odd–denominator fractional plateaus, it does not account for the vast sequence of other observed fractions such as 2/5, 3/7, or 4/9. This limitation ultimately motivated the development of a more general framework, *Composite Fermion theory.*

4 Composite Fermion theory

While the Laughlin wave-function beautifully explains the $\nu = 1/m$ series, it does not explain the other fractional hierarchies. A major breakthrough came with Jain's proposal that the essential physics of the FQHE can be understood by introducing new emergent quasiparticles called *composite fermions*.

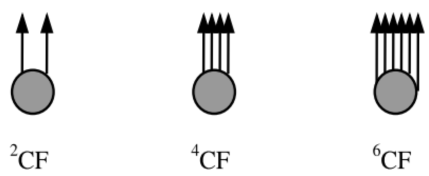


Figure 6: Figure shows different flavours of CFs. Each CF is an electron bounded with even number of flux quanta.

A composite fermion (CF) is formed when an electron binds an even number of quantized vortices (or equivalently, flux quanta) as shown in figure 6. A vortex, by definition, produces a phase of 2π for a closed path around it, which shows the topological nature of it. Importantly, the number of attached flux quanta must be *even* to maintain the fermionic nature of the composite particle, since the Aharonov–Bohm phase factor accumulates on electron due to exchange or half rotation around a flux quantum is $\exp(i\pi)$. The vortices carry phase winding, and by binding them to electrons, the effect of electron–electron repulsion is partially absorbed into an emergent gauge field. This transforms the original strongly interacting electron system into a system of weakly interacting composite fermions.

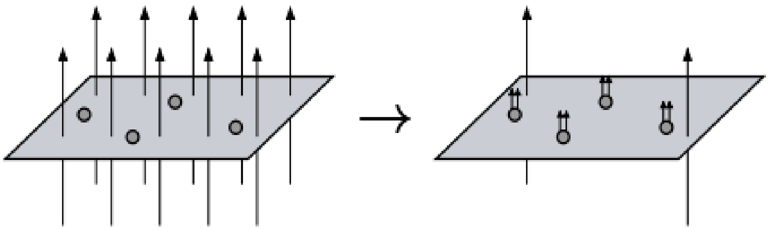


Figure 7: Right figure shows the CFs experienced reduced magnetic field compared to that of the interacting electrons in Left figure.

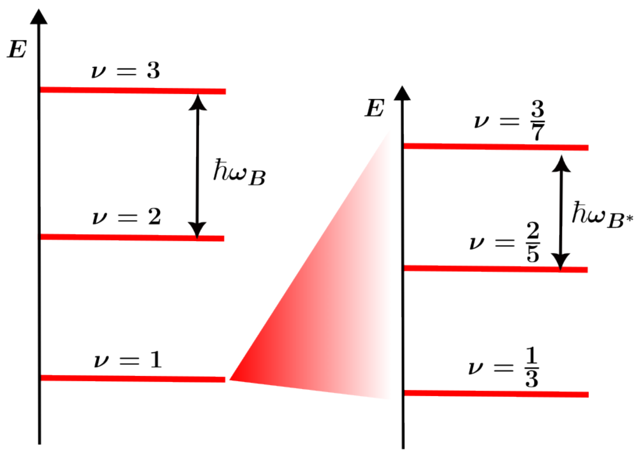


Figure 8: Left energy spectrum represents the Landau Levels in the external magnetic field B, while the right one shows the Λ Levels corresponding to LLL in reduced magnetic field B^*.

The key physical idea is that composite fermions experience a reduced effective magnetic field compared to electrons as shown in the figure 7, due to bunching of the external fluxes into CFs. Under this reduced field, they fill their own Landau levels, referred to as Λ–levels. For LLL filling, one can interpret the LLL is splitting to many Λ–levels as shown in the figure 8. Many fractional quantum Hall states turn out to be nothing more than the integer quantum Hall effect of composite fermions.

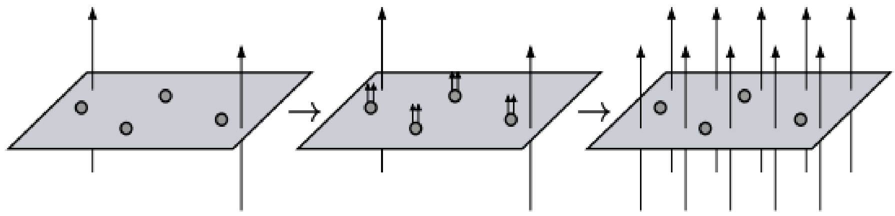


Figure 9: The flow diagram show systematic way to construct FQH state from IQH state through CF approach.

4.1 Mean–Field Approximation and Effective Magnetic Field

We demonstrate the Mean-field picture of composite fermion theory via the reverse way of the approach shown in figure 7. It begins by relating a strongly interacting electron system at filling ν to a weakly interacting CF system at an effective filling ν^*. Following the flow diagram of figure 9, we start with 2DEG of non-interacting electron of density ρ in a magnetic field $|B^*| = \rho\Phi_0/\nu^*$, which ensures the LL filling to be $\nu^* \in \mathbb{N}$. If each electron attaches $2p$ flux quanta ($p \in \mathbb{N}$) and becomes ^{2p}CF, the total additional emergent flux density becomes $2p\rho\Phi_0$. In the next step, under a mean–field treatment, this attached flux is smeared uniformly over the system, generating a total magnetic field

$$B = B^* + 2p\rho\Phi_0, \tag{4.1}$$

over the 2DEG. The crucial assumption on has to make is that there is no substantial qualitative change in energy spectrum of the system during the flux diffusion process as shown in the second step of figure 9. Notably, the value of $|B^*| = \rho\Phi_0/\nu^*$ implies that B is always positive. Thus in the mean-field picture, composite fermion in reduced magnetic field B^* is transformed into a strongly-interacting electron problem in higher magnetic field. This is the fundamental postulate of CF theory which states: *strongly interacting electrons turn into weakly interacting composite fermions*.

Now ν^* be the Landau level filling factor for non-interacting 2DEG such that:

$$\nu^* = \frac{\rho\Phi_0}{|B^*|} \in \mathbb{N}. \tag{4.2}$$

Converting the electrons to ^{2p}CFs, we start filling up the Λ levels with filling factor ν as shown in the figure 10. According to the fundamental postulate of CF theory, the aforementioned ν is also the filling factor of strongly interacting 2DEG in higher magnetic field B From the relation between the real and reduced magnetic fields, the electron filling factor becomes

$$\nu = \frac{\rho\Phi_0}{B} = \frac{\nu^*}{2p\nu^* \pm 1}. \tag{4.3}$$

This simple expression reproduces a wide array of experimentally observed fractions. For instance, taking $p = 1$ (which is for ^{2}CFs), and $\nu^* = 1, 2, 3, \ldots$ generates the sequence

$$\nu = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \ldots$$

which includes the most prominent FQH plateaus.

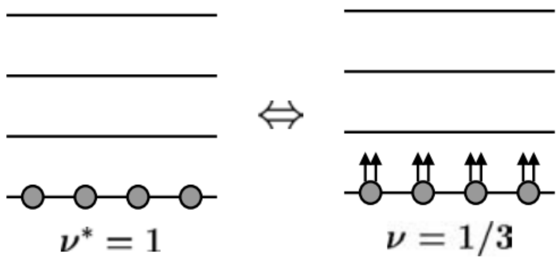


Figure 10: Left figure shows complete LLL filling in IQHE picture and the right figure shows corresponding complete lowest Λ level filling in CF picture

4.2 Berry Phase Interpretation

An alternative but equivalent viewpoint uses Berry phases. Consider there is a weakly interacting ^{2p}CF gas in 2D system of density ρ. Let a ^{2p}CF moving adiabatically in a closed loop encloses area A. The total phase acquired by the CF consists of two contributions: the usual Aharonov–Bohm phase from the external magnetic field, and the phase arising from encircling the attached $2p$ vortices bound to all other electrons inside the area. The total Berry phase is

$$\gamma = 2\pi \left(\frac{AB}{\Phi_0} - 2p\,\rho A \right). \tag{4.4}$$

Since a composite fermion behaved as a free charged particle in some effective magnetic field B^*, the Berry phase would instead be

$$\gamma = 2\pi \frac{AB^*}{\Phi_0}. \tag{4.5}$$

Equating the two expressions yields

$$B^* = B - 2p\,\rho\Phi_0, \tag{4.6}$$

the same effective magnetic field found from the mean–field argument. Using the definitions of ν and ν^* again gives the general relation

$$\nu = \frac{\nu^*}{2p\nu^* \pm 1}. \tag{4.7}$$

This Berry phase perspective gives an intuitive explanation that composite fermions move in reduced magnetic flux because each particle drags along its own flux tubes, partially canceling the external field.

4.3 New Filling Factors Generated by Composite Fermions

Using the expression

$$\nu = \frac{\nu^*}{2p\nu^* \pm 1}, \tag{4.8}$$

one can construct a hierarchy of FQHE filling factors. For example, taking $p = 1$ (i.e. attaching two flux quanta to each electron,and converting it into a ^{2}CF) gives:

ν^*	$\nu = \frac{\nu^*}{2\nu^* \pm 1}$
1	1/3, 1
2	2/5, 2/3
3	3/7, 3/5
4	4/9, 4/7
$\vdots$	$\vdots$

For $B^* > 0$ (i.e., parallel to CF flux), the real magnetic field is $B_> = B^* + 2p\,\rho\Phi_0$ and the observed filling fractions are

$$\nu_> = \frac{\nu^*}{2\nu^* + 1} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \dots \tag{4.9}$$

In similar prescription, for $B^* < 0$ (i.e., opposite of CF flux),the real magnetic field is $B_< = -|B^*| + 2p\,\rho\Phi_0$ and the observed filling fractions are

$$\nu_< = \frac{\nu^*}{2\nu^* - 1} = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \dots. \tag{4.10}$$

These sequences collectively explain most of the experimentally observed fractional quantum Hall states with odd denominators in the lowest Landau level. So all the fraction falls under a unified theory due to the assumption that the fractional quantum Hall effect of electrons is the integer quantum Hall effect of composite fermions.

5 Connection between CF theory and Laughlin's approach to FQHE

So far we have seen two approach to explain the FQHE. While Laughlin's approach is restricted to $\frac{1}{m}$ plateaus, CF theory provides vast hierarchies of ν than just $\frac{1}{m}$ which one can obtain by putting $\nu^* = 1$ and varying p. It is therefore necessary to investigate the connection between the Laughlin and CF approaches specifically at the filling fraction

$$\nu = \frac{1}{m} = \frac{1}{2p+1}, \quad m \in \mathbb{N}_{\text{odd}},\ p \in \mathbb{N}$$

5.1 Laughlin's Quasi-holes and Fractionalization

To establish the connection between the two theoretical frameworks, it is necessary to examine the quasi-hole excitation in the Laughlin wave-function, which manifests physically as an effective flux attach to the system. Laughlin's original construction already hinted at the presence of emergent quasiparticles with fractional charge and statistics. A quasi-hole

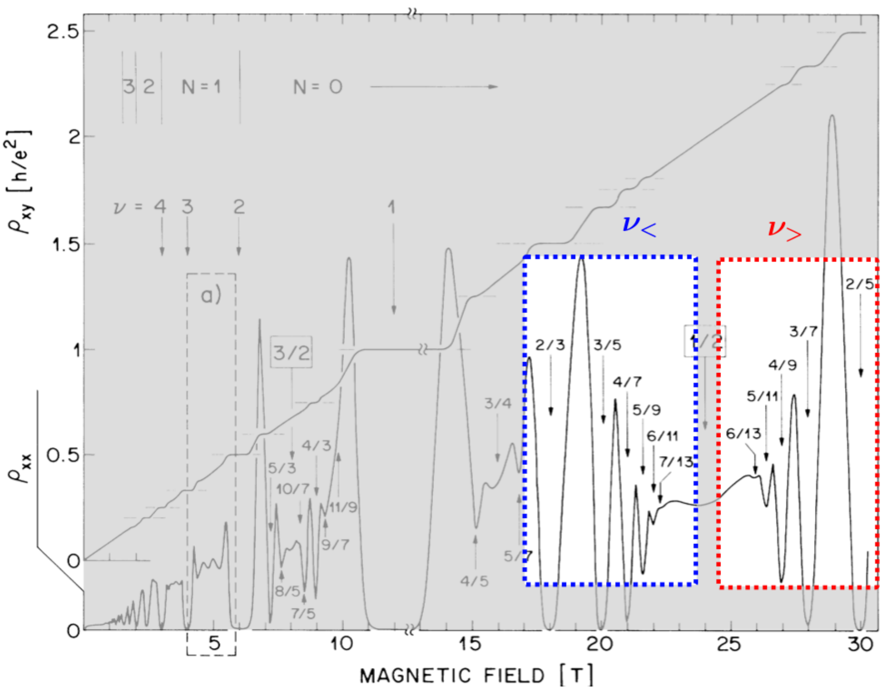


Figure 11: Figure shows the experimental Hall plateau of the $\nu_<$ and $\nu_>$ filling sequences at relatively lower $B_<$ and higher $B_>$ respectively.

located at position η is described by the Laughlin wave-function

$$\psi_m^{1-qh}(\{z_i\},\eta) = \prod_{i=1}^{N}(z_i-\eta)\prod_{i<j}(z_i-z_j)^m \exp\left(-\sum_{i=1}^{N}\frac{|z_i|^2}{4l_B^2}\right). \tag{5.1}$$

The additional factor $\prod_i(z_i-\eta)$ introduces a phase winding around the point η and effectively removes electronic density and creating a local deficit of electron, i.e., a *quasi-hole*. Similarly multiple quasi-holes at $\{\eta_j\}$ are described by the wave function

$$\psi_m^{M-qh}(\{z_i\},\{\eta_j\}) = \prod_{j=1}^{M}\prod_{i=1}^{N}(z_i-\eta_j)\prod_{i<j}(z_i-z_j)^m \exp\left(-\sum_{i=1}^{N}\frac{|z_i|^2}{4l_B^2}\right). \tag{5.2}$$

Two striking properties of the quasi-hole states are charge fractionalization and fractional braiding statistics. When a single quasi-hole state ψ_m^{1-qh} is adiabatically evolved such a way that the quasi-hole encircles a magnetic flux Φ, the accumulated Berry phase is

$$\gamma = \frac{e}{m}\frac{\Phi}{h} = e^*\frac{\Phi}{h} \tag{5.3}$$

where $e^* = e/m$ is effective fractional charge. Comparing to the standard result of Berry phase accumulation associated to a charged particle, one can see that the charge of the quasi-hole is fractionalized. Furthermore, exchanging two quasi-holes adiabatically in ψ_m^{2-qh} produces a phase factor

$$\exp\left(\iota\, 2\pi \frac{1}{m}\right), \tag{5.4}$$

indicating that Laughlin quasi-holes are *anyons*. This was the indication that fractionalized excitations naturally emerge from strongly correlated quantum fluids.

5.2 Connection through Spectral Flow in the Corbino Geometry

The link between flux insertion and quasi-hole creation becomes especially clear in the Corbino disc geometry. As we have seen in earlier discussion that adiabatically inserting one flux quantum at the centre ($z = 0$) of the Corbino disc shifts every LLL orbitals, which essentially increase the angular momentum of all the states by unity. The wave-functions of which is achieved by multiplying the old wave-function by the factor $\prod_{i=1}^{N} z_i$, i.e.,

$$\psi_m(z) \mapsto \psi_{m+1}(z) = \left(\prod_{i=1}^{N} z_i\right) \psi_m(z). \tag{5.5}$$

This extra factor $\prod_i z_i$ is exactly what appears in the quasi-hole wave-function when the hole is created at $\eta = 0$ i.e.,

$$\psi_m^{1-qh}(\{z_i\}, 0) = \prod_{i=1}^{N} z_i \prod_{i<j} (z_i - z_j)^m e^{-\sum |z_i|^2/4l_B^2} = \left(\prod_{i=1}^{N} z_i\right) \psi_{LLL}(\{z_i\}). \tag{5.6}$$

One can reduce the inner circle of the Corbino disc to vanishingly small and the inserted solenoid become vanishingly small. Thus, inserting a flux quantum at the centre is equivalent to creating a quasi-hole at that point. Consequently spectral flow due single flux quantum insertion will create the charge deficiency of a quasi-hole in the centre, which is fractionalized charge $e^* = e/m$. Hall conductivity associated with it is,

$$\sigma_{xy} = \frac{e^*}{\Phi_0} = \frac{1}{m} \frac{e}{\Phi_0}, \tag{5.7}$$

which is consistent with the fractional filling of $\nu = 1/m$.

5.3 Laughlin Wave-function as a Composite Fermion State

The final piece of puzzle is that the Laughlin wave-function can be naturally interpreted within the composite fermion picture. Starting from Laughlin wave-function

$$\psi_m(\{z_i\}) = \prod_{i<j} (z_i - z_j)^m \, e^{-\sum |z_i|^2/4l_B^2}, \tag{5.8}$$

we rewrite it as

$$\psi_m(\{z_i\}) = \prod_{i<j} (z_i - z_j)^{m-1} \times \prod_{i<j} (z_i - z_j)\, e^{-\sum |z_i|^2/(4l_B^2)}. \tag{5.9}$$

The last factor

$$\prod_{i<j}(z_i - z_j)\, e^{-\sum |z_i|^2/(4l_B^2)} = \psi_{LLL}(\{z_i\}) \tag{5.10}$$

is precisely the non-interacting many–body ground state of a filled LLL in the IQHE. Following the arguments outlined above, the additional factor $(z_i - z_j)^{m-1}$ can be interpreted as attaching $(m-1)$ even number of flux quanta to each electron. Thus each non-interacting electron sees a flux quanta of strength $(m-1)\Phi_0$ on every other electrons which is exactly the definition of a composite fermion. Moreover the solution of mean-field Hamiltonian for composite fermion ${}^{m-1}$CF is exactly the Laughlin wave-function of $\nu = 1/m$. This unifies the Laughlin approach with the composite fermion model and shows that the Laughlin wave-function is the simplest example within the broader composite fermion framework.

6 Conclusion

The composite fermion picture provides a remarkably simple bridge between the integer and fractional quantum Hall effects. In the IQHE, non–interacting electrons in a magnetic field fill Landau levels, leading to integer quantized Hall conductance. In the FQHE, electrons are strongly interacting, yet many fractional states can be re–interpreted as integer quantum Hall states of composite fermions. This correspondence becomes clearer when one compare the three descriptions side– by–side as in 1. Composite fermions behave like non–interacting particles in an effective field B^*, whereas electrons in the real system experience the full magnetic field B and strong interactions.It thus provide the "middle layer" that connects the simple single particle physics of IQHE to the highly correlated FQHE. Many fractional filling factors correspond exactly to integer fillings ν^* for composite fermions in the reduced field B^*.

FQHE	CF	IQHE
Strongly interacting electrons	Weakly interacting CFs	Non–interacting electrons
Real field B	$B^* = B - 2pn\Phi_0$	Reduced field B^*
Λ Levels	Λ Levels	Landau levels
$\nu \in \mathbb{Q}_{>0}$	$\nu \in \mathbb{Q}_{>0}$	$\nu^* \in \mathbb{N}$

Table 1: Comparison among associated elements of FQHE, CFs, and IQHE

In this article, we began with an analysis of the Integer Quantum Hall Effect (IQHE) before transitioning to a discussion of the Fractional Quantum Hall Effect (FQHE) using both the Laughlin approach and the Composite Fermion (CF) theory. Further for the common ground of filling fraction $\nu = 1/m$, where both the theories - Laughlin's approach and CF theory are valid, we have shown the strong connection, which further validate the CF model.

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